Topic - Integration of Rational Algebraic Functions

Class - B.Sc (Hons)

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(I) Integration by Partial fraction

If $F(x) = \frac{g(x)}{\psi(x)}$

we break $\frac{g(x)}{\psi(x)}$ into smaller fractions

for this degree of polynomial $g(x)$ must be less than degree of polynomial $\psi(x)$.

If the Degree of polynomial $g(x) \geq$ Degree of polynomial $\psi(x)$.

In this cases we divide $g(x)$ by $\psi(x)$ to get

$$\frac{g(x)}{\psi(x)} = \text{Quotient} + \frac{\text{Remainder}}{\psi(x)}$$

In which fractional part meets the necessary requirements.

Problem 1. Integrate

$$\int \frac{x^2+1}{x(x^2-1)} dx$$

Let $I = \int \frac{x^2+1}{x(x^2-1)} dx$

Here Degree of polynomial N^r < Degree of polynomial D^r.

$$= \int \frac{x^2+1}{x(x+1)(x-1)} dx$$

Let $\frac{x^2+1}{x(x+1)(x-1)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1}$ ——— (1)

$$I = \int \frac{x^2+1}{x(x+1)(x-1)} dx = A\int \frac{dx}{x} + B\int \frac{dx}{x+1} + C\int \frac{dx}{x-1}$$

$$= A\log x + B\log(x+1) + C\log(x-1) + 7-($$

Now we have to calculate A, B, C

From (1)

$$\frac{x^2+1}{x(x+1)(x-1)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1}$$

$$= \frac{A(x^2-1) + B(x-1)x + Cx(x+1)}{x(x+1)(x-1)}$$

$\Rightarrow$ $A(x^2-1) + B(x-1)x + C\,x(x+1) = x^2+1$

put $x = 1$

$C \cdot 1 \cdot 2 = 2$

$C = 1$

Put $x = -1$

$+2B = 2$

$\Rightarrow B = 1$

Put $x = 0$

$-A - B + 0 = 1$

$A = -1$

Putting the value of A, B, C in (1)

$$I = -\log x + \log(x+1) + \log(x-1) + \lambda.$$

$$= \log \frac{(x+1)(x-1)}{x} + \lambda$$

$$= \log \frac{x^2-1}{x} + \lambda.$$ Here λ = Constant of Integration

<u>Problem 2</u>. Integrate $\int \frac{x^2+x+2}{(x-2)(x-1)} dx$

Let $I = \int \frac{x^2+x+2}{(x-2)(x-1)} dx$.

Here Degree of Polynomial of N^r
= Degree of Polynomial of D^r.

So we must have to divide N^r by D^r

$D^r = x^2 - 3x + 2$

$N^r = x^2 + x + 2$

$$x^2-3x+2 \overline{)\, x^2+x+2 \,(} 1$$
$$\underline{\cancel{x^2} - 3x + 2}$$
$$4x.$$

$$\therefore \frac{N^r}{D^r} = \frac{(x^2-3x+2)\cdot 1 + 4x}{x^2-3x+2}$$

$$= 1 + 4 \cdot \frac{x}{x^2-3x+2}$$

$$= 1 + 4 \cdot \frac{x}{(x-2)(x-1)}$$

Now $I = \int dx + 4 \int \frac{x}{(x-2)(x-1)} dx$

$= I_1 + 4 I_2$ —— (1)

Now $I_2 = \frac{x}{(x-2)(x-1)} = \frac{A}{x-2} + \frac{B}{x-1}$ —— (II)

$$I_2 = \int \frac{x}{(x-2)(x-1)} dx = A \int \frac{dx}{x-2} + B \int \frac{dx}{x-1}$$

$$= A \log(x-2) + B \log(x-1) \quad \text{—— (III)}$$

From (II)

$$\frac{x}{(x-2)(x-1)} = \frac{A}{x-2} + \frac{B}{x-1}$$

$$x = A(x-1) + B(x-2)$$

put $x = 1$

$$1 = \cancel{A} - B \Rightarrow B = -1$$

put $x = 2$

$$2 = A$$

putting these values of A and B in (III)

$$I_2 = 2\log(x-2) - \log(x-1)$$

$$= \log \frac{(x-2)^2}{x-1}$$

$$I_1 = \int dx = x$$

$\therefore$ From (I)

$$I = I_1 + 4 I_2 + \lambda.$$

$$= x + 4 \log \frac{(x-2)^2}{x-1} + \lambda. \quad \text{Where } \lambda = \text{constant of integration}$$

Problem 3. Integrals

$$\int \frac{x^3}{x^2+8x+12}\,dx.$$

Let $I = \int \frac{x^3}{x^2+8x+12}\,dx$

Here Degree of polynomial in N^r > Degree of polynomial in N^r.

Thus we divide N^r by D^r

$$\frac{x^3}{x^2+8x+12} = \text{quotient} + \frac{\text{Remainder}}{\text{Divisor}}$$

$$\begin{array}{r|l}
x^2+8x+12 \;) & x^3 \quad (x-8 \\
 & \cancel{x^3} \pm 8x^2 \pm 12x \\ \hline
 & -\cancel{8x^2} - 12x \\
 & -8x^2 - 64x - 96 \\
 & \;\;+ \quad\;\; + \quad\;\; + \\ \hline
 & 52x + 96
\end{array}$$

$$= x-8 + \frac{52x+96}{x^2+8x+12}$$

$$= x-8 + \frac{52x+96}{(x+2)(x+6)}$$

$$\therefore I = \int (x-8)\,dx + \int \frac{52x+96}{(x+2)(x+6)}\,dx$$

$$= I_1 + I_2 \qquad \text{(i)}$$

Now $I_1 = \int (x-8)\,dx = \frac{x^2}{2} - 8x.$ (ii)

$$I_2 = \int \frac{52x+96}{(x+2)(x+6)}\,dx$$

let $\dfrac{52x+96}{(x+2)(x+6)} = \dfrac{A}{x+2} + \dfrac{B}{x+6}$ —— (III)

$\therefore \displaystyle\int \frac{52x+96}{(x+2)(x+6)}\,dx = A\int \frac{dx}{x+2} + B\int \frac{dx}{x+6}$ —— (IV)

$= A\log(x+2) + B\log(x+6)$.

for A and B.

from (I) $\dfrac{52x+96}{(x+2)(x+6)} = \dfrac{A(x+6)+B(x+2)}{(x+2)(x+6)}$

$\Rightarrow 52x+96 = A(x+6)+B(x+2)$

put $x=-6$.

$-312+96 = -4B$

$-216 = -4B$

$B = 54$

put $x=-2$

$-104+96 = 4A$

$-8 = 4A$

$\Rightarrow A = -2$

$\therefore$ from (IV)

$$\int \frac{52x+96}{(x+2)(x+6)}\,dx = -2\log(x+2) + 54\log(x+6)$$

putting these value in (I)

$I = \frac{x^2}{2} - 8x - 2\log(x+2) + 54\log(x+6) + \lambda$

which is the required integral